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Field theory models for tachyon and gauge field string dynamics

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Abstract: In hep-th/0008227, the unstable lump solution of ϕ^3 theory was shown to have a spectrum governed by the solvable Schroedinger equation with the $\ell = 3$ reflectionless potential and was used as a model for tachyon condensation in string theory. In this paper we study in detail an $\ell \to \infty$ scalar field theory model whose lump solution mimics remarkably the string theory setup: the original field theory tachyon and the lump tachyon have the same mass, the spectrum of the lump consists of equally spaced infinite levels, there is no continuous spectrum, and nothing survives after tachyon condensation. We also find exact solutions for lumps with codimension ≥ 2, and show that that their tensions satisfy $\frac{1}{2\pi}\frac{T_p}{T_{p+1}} = \frac{e}{\sqrt{2\pi}} \approx 1.08$. We incorporate gauge fixed couplings to a U(1) gauge field which preserve solvability and result in massless gauge fields on the lump.

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1. Introduction and summary

While the precise description of the physics of tachyon condensation and D-brane annihilation may require the complete framework of string field theory, it is of interest to consider simpler models to understand some of the puzzling issues that arise. Indeed, tachyon condensation is simpler in the case of p-adic open string theory [1], and in the case when large non-commutativity is introduced on the D-branes [2]–[7].

In a recent paper [8] a model was developed where, just as in string field theory, we can explore the decay of a soliton as seen by the field theory living on the world-volume of the soliton itself. The model is based on the unstable lump solution of the simple ϕ^3 theory, the truncation of open string field theory to the tachyon field only (this lump was first studied numerically in [9]). Not only is the exact lump profile readily written in terms of hyperbolic functions, but the full spectrum of the field theory on the lump world-volume is readily obtained. This happens because the potential in the relevant Schroedinger type equation is the $\ell = 3$ case of the infinite series of exactly solvable reflectionless potentials:[1] $U_\ell(x) = -\ell\,(\ell+1)\,\mathrm{sech}^2\,x$. The potential U_ℓ gives precisely ℓ bound states, and a continuum, whose lowest energy state is a bound state at threshold. The potentials U_1 and U_2 were known to be relevant to the sine-Gordon soliton and the ϕ^4 kink, respectively [12]. In both cases the wavefunction of the lowest energy bound state in the quantum mechanical problem is identified with the derivative of the soliton profile. This procedure does not give a simple result for $\ell \geq 3$ [12, 13]. In the model of [8] it is the middle bound state of the U_3 potential that is identified with the derivative of the lump profile. The lowest bound state then represents the tachyonic fluctuation around the lump. Motivated by this result, one is led naturally to consider building lumps for $\ell > 3$ by using the next to lowest bound state of U_ℓ to define the profile derivative. The scalar field theory potentials giving rise to such unstable lumps were found recently [14]. The remarkably simple field theory potentials $V_{\ell+1}(\phi)$ (whose lumps are governed by $U_{\ell+1}$) consist of two terms, a ϕ^2 term and a $(\phi^2)^{1+1/\ell}$ term. For each $V_{\ell+1}$ the lump worldvolume theory will have one tachyon, one massless scalar, $(\ell-1)$ massive scalars and a continuum spectrum.

In this paper we study these field theories as models for tachyon condensation on unstable branes. Our focus, however, will be in the model obtained as an $\ell \to \infty$ limit. As will be explained in the text, by simultaneously letting $\ell \to \infty$ and rescaling the spacetime coordinates, one obtains the well-defined potential

$$V_\infty(\phi) = -\frac{1}{4}\phi^2 \ln \phi^2\,. \tag{1.1}$$

This very unusual potential has an unstable critical point at $\phi_0^2 = e^{-1}$, with a tachyon mass squared $m^2 = -1$. Note, in addition, the local minimum at $\phi = 0$, where $V'' \to +\infty$ and the scalar field acquires infinite mass. With a standard kinetic term for ϕ we have a simple field theory of a tachyon with the expected property that as the tachyon condenses to the local minimum it disappears from the spectrum by acquiring an infinite mass.

The lump solution in this field theory is a simple gaussian $\overline{\phi}(x) = e^{-x^2/4}$ and the Schroedinger potential U_∞ is that of the one-dimensional simple harmonic oscillator! The spectrum of the lump field theory includes a tachyon, a massless field, and an infinite tower of equally spaced massive scalars. It is noteworthy that the tachyon living on the lump has $m^2 = -1$, just as the mass of the tachyon in the field theory model giving rise to the lump. This is true for string theory D-branes and for p-adic solitonic q-branes [1] but not for the finite ℓ models. Perhaps more significant is the fact that the continuum spectrum of the finite ℓ models has disappeared. Indeed, in ordinary field theory, the continuous spectrum of the soliton is a reflection of the

[1] For a pedagogical review on these and other solvable hamiltonians, with references to the early literature see [10]. Applications of reflectionless systems to fermions can be found in [11].

degrees of freedom that exist on the vacuum defined by the asymptotic field expectation value of the soliton at spatial infinity. For ordinary potentials, the asymptotic field configuration is a vacuum with perturbative particle excitations. For the case of (1.1) the lump is asymptotic to the $\phi = 0$ configuration, around which there is no perturbative dynamics.

Tachyon condensation on the lump is easily studied. The above theory has the special feature that the wavefunction representing the tachyon mode on the lump is actually proportional to the lump profile itself. This means that out of the infinite number of fields living on the lump, only the tachyon needs to flow in order to destroy the lump. It can be seen explicitly how this flow makes all fields on the lump flow to infinite mass.

The massless mode on the lump is just a derivative of the lump profile. Hence, if this mode is given a small expectation value, then the lump will have a correspondingly small shift in position. This mode behaves as a marginal operator in string field theory. Just as in the $\ell = 3$ model, we confirm that in the $\ell = \infty$ model the relation between displacement and the marginal parameter is two to one. Both small displacements and large displacements lead to a small marginal parameter, with the parameter taking a finite maximum value in between. In the present model, however, we are able to see explicitly how higher level states help realize large displacements; for a large shift x_0, the largest contribution to the shift comes from modes with level number L satisfying $L \sim x_0^2/4$. These results provide evidence for the missing branch proposal of [8] which suggests that a different choice of branch for higher level fields describes large marginal deformations in string field theory. The branch point occurs at the maximal value for the string field marginal parameter [15].

There is another string theory parallel. For the $\ell = \infty$ model we can find the profiles of all lower codimension lump solutions explicitly. Thought of as branes of various spatial dimensions, we find that the tensions satisfy $\frac{1}{2\pi}\frac{T_p}{T_{p+1}} = \frac{e}{\sqrt{2\pi}} (\simeq 1.084)$, for all possible p. In string theory this ratio is unity. A similar property was found to hold in p-adic string theory, where the above ratio depends on the prime number used to define the model.

We also extend the models to incorporate a U(1) gauge field. The terms we add were suggested by the form of level expanded string theory, but our main constraints come from requiring: (i) solvability of the spectrum, (ii) inheriting an exactly massless gauge field on the worlvolume of the lump and (iii) existance of the model for all finite ℓ. These conditions can all be satisfied and lead to some special forms for scalar-gauge field interactions. It is possible, in addition, to guarantee that the gauge field components transverse to the brane do not give rise to additional massless states. This is required because the translation mode of the brane arises from the tachyon field. The lagrangians we write should be thought as gauge fixed ones, just as level expanded versions of string theory are generally worked out in the Siegel gauge. We do not know if there is a gauge invariant version of the models.

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This paper is organized as follows. In section 2 we introduce the field theory scalar models associated with arbitrary ℓ and infinite ℓ. We also calculate the spectrum of the lump worldvolume field theories in both cases. In section 3 we show explicitly how in the $\ell = \infty$ model tachyon condensation sends the masses of all fields living on the lump to infinity. In section 4 we discuss large displacements of $\ell = \infty$ lumps, and study an effective field theory approximation where we include only the tachyon and the massless field on the lump. In section 5 we study higher codimension lumps, give their exact profiles and calculate their tensions. In section 6 we show how to incorporate gauge fields into the models. Some concluding remarks are offered in section 7.

2. Finite ℓ models and the $\ell \to \infty$ model

In this section we construct a series of field theory models whose lump profiles are exactly solvable. Moreover, the spectrum of fluctuations about the lumps is also solvable. In the first subsection, we review a construction of Goldstone and Jaffe [14] for building field theory potentials giving rise to lumps with spectra controlled by the $U_{\ell+1}$ reflectionless potentials. We then construct a new potential by taking $\ell \to \infty$, and investigate various properties of this potential. In the second subsection, we discuss the spectrum of lump fluctuations.

2.1 Introducing the models

As our starting point, let us consider the one-dimensional Schroedinger equation for the $U_{\ell+1}$ potential,

$$-\frac{d^2\psi(x)}{dx^2} - (\ell+1)(\ell+2)\,\mathrm{sech}^2(x)\psi(x) = E\,\psi(x)\,. \tag{2.1}$$

If ℓ is an integer, then the potential is reflectionless, in other words a wave traveling in from the right will be completely transmitted when crossing over the potential. This Schroedinger equation in (2.1) is completely solvable. The ground state wave function has the solution

$$\psi_0(x) = \frac{1}{\cosh^{\ell+1}(x)}\,, \tag{2.2}$$

with energy

$$E_0 = -(\ell+1)^2\,. \tag{2.3}$$

There are a finite number of bound solutions, whose wavefunctions are the ground state wavefunction in (2.2) multiplied by Hermite polynomials in $\sinh(x)$. The energy levels for these bound states are

$$E_n = -(\ell+1-n)^2\,, \quad 0 \le n < \ell+1\,. \tag{2.4}$$

Notice that $n = \ell+1$ would give zero energy. Indeed at the bottom of the continuum there is a bound state at threshold. This state will not play an important role in all that follows.

The field theory potential giving rise to a lump solution with fluctuation spectrum governed by the Schroedinger equation (2.1) was found in [14][2]. It is given by

$$V_{\ell+1}(\phi) = \frac{\ell}{4}\phi^2\left(1-\phi^{2/\ell}\right). \tag{2.5}$$

For $\ell > 2$ we define $\phi^{2/\ell} \equiv (\phi^2)^{1/\ell}$ with the $\ell^{\rm th}$ root real and positive.

Let us review the construction of the potential in (2.5). We assume the following lagrangian for the field theory:

$$\mathcal{L}_{\ell+1} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_{\ell+1}(\phi)\,, \tag{2.6}$$

where $V_{\ell+1}(\phi)$ is to be determined, and we work with the metric $(-,+\cdots+)$. A lump solution to the equations of motion satisfies

$$\overline{\phi}\,''(x) = {V_{\ell+1}}'(\overline{\phi}(x))\,, \tag{2.7}$$

which, after integrating is

$$\frac{1}{2}(\overline{\phi}\,'(x))^2 = V_{\ell+1}(\overline{\phi}(x))\,. \tag{2.8}$$

An integration constant has been absorbed into the definition of $V_{\ell+1}(\phi)$, making the potential vanish at the asymptotic value taken by the field in the lump solution. Now assume that the *derivative* of $\overline{\phi}(x)$ is proportional to the first excited state wavefunction of (2.1),

$$\overline{\phi}'(x) = -\sqrt{\frac{\ell}{2}}\,\frac{\sinh(x/\sqrt{2\ell})}{\cosh^{\ell+1}(x/\sqrt{2\ell})}\,, \tag{2.9}$$

where the argument of the cosh and sinh functions has been rescaled for later convenience. The derivative acting on $\overline{\phi}$ generates a translation of the lump, hence the wavefunction in (2.9) is a zero mode of the fluctuation spectrum. The expression in (2.9) is easily integrated, giving

$$\overline{\phi}(x) = \frac{1}{\cosh^{\ell}(x/\sqrt{2\ell})}\,. \tag{2.10}$$

We can now use standard identities to express $(\overline{\phi}'(x))^2$ in terms of $\overline{\phi}$,

$$(\overline{\phi}\,')^2 = \frac{\ell}{2}\overline{\phi}^{\,2}\left(1-\overline{\phi}^{\,2/\ell}\right). \tag{2.11}$$

Hence, using (2.8) the potential is given by (2.5).

[2]We thank J. Goldstone and R. Jaffe for informing us of this result prior to publication.

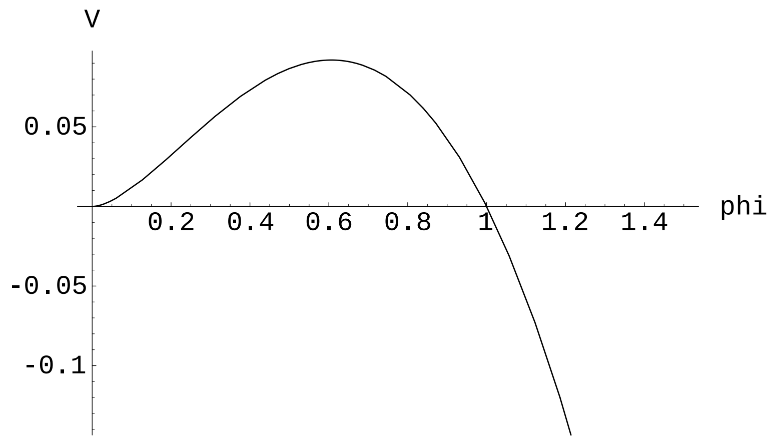

Figure 1: The $\ell \to \infty$ potential $V_\infty(\phi) = -\frac{1}{4}\phi^2 \ln \phi^2$.

The potential is not bounded from below, thus we expect the fluctuations about the lumps to have tachyonic modes. $V_2(\phi)$ is an inverted quartic potential. $V_3(\phi)$ is the ϕ^3 potential that describes the level zero string field theory and the model studied in detail in [8]. For $\ell > 2$, the potential $V_{\ell+1}(\phi)$ is non polynomial.

The potential $V_{\ell+1}(\phi)$ has a maximum at

$$\phi = \phi_0 = \left(\frac{\ell}{\ell+1}\right)^{\ell/2}, \tag{2.12}$$

and a local minimum at $\phi = 0$. At the maximum, the second derivative of the potential is given by

$$V''(\phi_0) = -1\,, \tag{2.13}$$

while at the minimum, it is

$$V''(0) = \frac{\ell}{2}\,. \tag{2.14}$$

Given the general behavior of these potentials, it is tempting to use one of the higher ℓ models as a toy model of open string field theory. The analog of the open string vacuum is the maximum $\phi = \phi_0$, with an open string tachyon of mass squared $m^2 = -1$. The closed string vacuum analog is the local minimum at $\phi = 0$.

At the local minimum, the scalar field has a mass squared $m^2 = \ell/2$, hence in the limit $\ell \to \infty$, the scalar field decouples. However, the tachyon mass at the maximum remains finite. Hence, the $\ell \to \infty$ model captures some of the salient features of open string field theory. In this limit, the potential in (2.5) simplifies to

$$V_\infty(\phi) = -\frac{1}{4}\phi^2 \ln \phi^2\,, \tag{2.15}$$

which has a maximum at $\phi_0 = e^{-1/2}$. This potential is shown in figure 1. Its higher

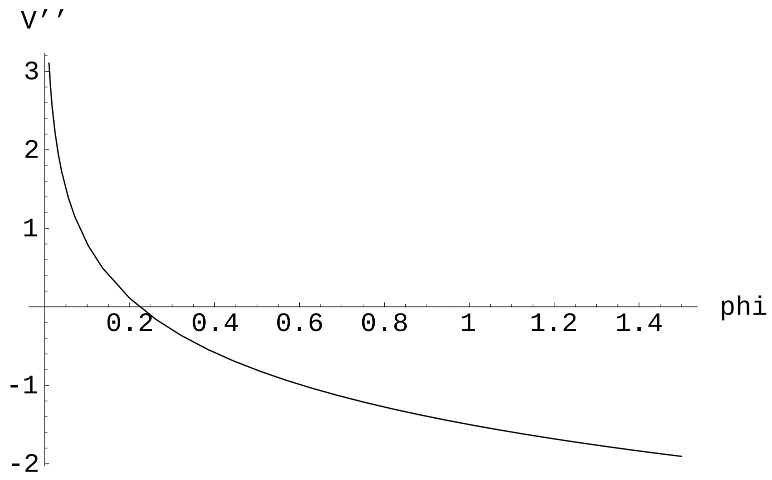

Figure 2: The second derivative ${V_\infty}''(\phi) = -\frac{1}{2}(3 + \ln\phi^2)$.

derivatives are also of interest. In particular, note that

$$
{V_\infty}'' = -\frac{1}{2}\ln\phi^2 - \frac{3}{2}\,, \tag{2.16}
$$

which blows up as $\phi \to 0$. Hence the effective scalar mass blows up as ϕ flows to the analog of the closed string vacuum. This is shown in figure 2.

We can easily find the lump solution for the $\ell \to \infty$ case by solving (2.8) by quadratures. This gives

$$
x = \int_{\overline{\phi}}^{1} \frac{d\phi}{\sqrt{2V_\infty(\phi)}} = 2\sqrt{-\ln\overline{\phi}} \longrightarrow \overline{\phi}(x) = \exp\left(-\frac{x^2}{4}\right), \tag{2.17}
$$

a simple gaussian. This is reminiscent of the lump solutions for the p-adic string field theory [1].

2.2 Field theory spectrum on the lumps

We now consider the spectrum for the codimension one lumps of these models, both for finite ℓ and in the $\ell \to \infty$ limit. If we expand ϕ about the lump solution $\overline{\phi}$, then the fluctuation modes satisfy the Schroedinger equation

$$
-\frac{d^2\,\psi(x)}{dx^2} + {V_{\ell+1}}''(\overline{\phi}(x))\psi(x) = m^2\psi(x)\,. \tag{2.18}
$$

Substituting (2.5) and (2.10) into (2.18), the equation becomes

$$
-\frac{d^2}{dx^2}\psi(x) + \frac{1}{2\ell}\left(\ell^2 - (\ell+1)(\ell+2)\,\mathrm{sech}^2\left(\frac{x}{\sqrt{2\ell}}\right)\right)\psi(x) = m^2\psi(x)\,. \tag{2.19}
$$

Except for the shift and rescaling of the potential, and the corresponding rescaling of x, this is the reflectionless potential in (2.1). We can then find the masses by shifting and rescaling (2.4), leaving us with

$$\begin{aligned} m^2 &= \frac{1}{2\ell}\left(\ell^2 - (\ell+1-n)^2\right), \quad 0 \le n < \ell+1 \\ &= \frac{(n-1)(2\ell-n+1)}{2\ell}. \end{aligned} \tag{2.20}$$

From (2.20) we see that the lowest mode has negative $m^2 = -1 - 1/2\ell$ and that the next mode is massless. We also see that there are $\ell - 1$ discrete massive modes and a continuous spectrum starting at $m^2 = \ell/2$.

If we take the $\ell \to \infty$ limit, then (2.20) reduces to $m^2 = n-1$ and the continuum is pushed to infinite mass. We can see this explicitly by inserting the potential $V_\infty(\phi)$ in (2.15) and the lump solution in (2.17) into (2.18). Then the Schroedinger equation for the fluctuation modes reduces to the harmonic oscillator equation

$$-\frac{d^2}{dx^2}\psi(x) + \left(-\frac{3}{2} + \frac{1}{4}x^2\right)\psi(x) = m^2\psi(x). \tag{2.21}$$

The masses of the modes are well known and satisfy

$$m^2 = n - 1, \quad n \ge 0. \tag{2.22}$$

Hence, unlike the finite ℓ models, the $\ell \to \infty$ model shares three important properties with string theory. First, the fluctuations about the lumps have only a discrete spectrum. Second, the mass of the tachyon fluctuation equals the mass of the tachyon in the original field theory. Third, there is equal spacing between levels. Since in string theory we assign level zero to the tachyon we do the same here and define the level $L(n)$ to be

$$L(n) = n. \tag{2.23}$$

The lump solutions have a profile in which the field falls off to the closed string vacuum as $|x| \to \infty$. Examining (2.18), we see that a necessary condition for a discrete spectrum is that $V'' \to \infty$ as $|x| \to \infty$. But this is simply the statement that the scalar mass becomes infinite as the scalar field descends towards the analog of the closed string vacuum. In fact, the discrete spectrum condition should be true for models that include higher modes in the level expansion, and even the complete open string field theory. Hence, the decoupling of the open string tachyon as a propagating mode is consistent with the fact that the open string spectrum of D-branes is discrete.

3. Tachyon condensation and mass flow at $\ell = \infty$

In this section we explore how the lumps decay through tachyon condensation. In the limit that $\ell \to \infty$, the process is especially simple.

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In the previous section, we learned that the finite ℓ models have lump profiles of the form $\overline{\phi}(x) = \text{sech}^{\ell}(x/\sqrt{2\ell})$. However, the tachyon fluctuation of the lump has a wavefunction $\psi_0(x) = \text{sech}^{\ell+1}(x/\sqrt{2\ell})$. When the lump decays, the fluctuations become large, such that the lump profile is exactly cancelled by the fluctuations. However, since the profile wavefunction is not quite the same as the tachyon fluctuation, other modes must be turned on in order to annihilate the lump. This was explored in great detail for the $\ell = 2$ case in [8].

However, when ℓ approaches infinity, the tachyon wavefunction is the lump profile. We can see this explicitly from the $\ell \to \infty$ Schroedinger equation in (2.21). Since this is the equation for a harmonic oscillator, the ground state wavefunction *is* the gaussian $\psi_0(x) = e^{-x^2/4}$, which is the lump profile equation. Hence, when the lump decays, only the tachyon mode condenses.

Let us show more explicitly that the tachyon flow does not induce condensation for the other modes. To this end, write the scalar field ϕ as

$$\phi = (1+T)\,\overline{\phi}(x) + \psi(x) = (1+T)\,e^{-x^2/4} + \psi(x)\,, \tag{3.1}$$

where the first term is an expectation value, and $\psi(x)$ denotes fluctuations about it. The lump has decayed when T reaches $T = -1$. If the other fields do not condense under the flow, then their one point functions in the effective lagrangian must be seen to vanish. Expanding the action corresponding to (2.6) with $\ell = \infty$ about the decaying lump described in (3.1) and using the equations of motion, the one point contributions are

$$\begin{aligned} &\int dx \Big[-(1+T)\partial_x{}^2\overline{\phi}(x) + V_\infty{}'((1+T)\overline{\phi}(x))\Big]\psi(x) = \\ &\qquad = \int dx \Big[-(1+T)V_\infty{}'(\overline{\phi}(x)) + V_\infty{}'((1+T)\overline{\phi}(x))\Big]\psi(x)\,, \end{aligned} \tag{3.2}$$

where $\psi(x)$ includes all fluctuation modes. Using the explicit form for $V_\infty(\phi)$ in (2.15), we find that the term inside the square brackets is $-\overline{\phi}(x)\ln(1+T)$. Since $\overline{\phi}(x)$ is proportional to the ground state wave function and except for its tachyon component $\psi(x)$ is orthogonal to the ground state, the integral over x is zero. Hence the one point functions of all modes other than the tachyon are zero.

Next, let us examine what happens to the masses as the tachyon condenses. The effective mass for the modes is found by solving the eigenvalue equation

$$-\frac{d^2\,\psi(x)}{dx^2} + V_\infty{}''\left((1+T)\overline{\phi}(x)\right)\psi(x) = m^2\psi(x)\,. \tag{3.3}$$

Using the relation

$$V_\infty{}''\left((1+T)\overline{\phi}(x)\right) = V_\infty{}''(\overline{\phi}) - \ln(1+T)\,, \tag{3.4}$$

we see that (3.3) is the same as (2.21) with a finite shift in the mass, $\Delta m^2 = -\ln(1+T)$. As $T \to -1$, this shift blows up and the modes become infinitely massive, effectively decoupling from the spectrum.

4. Transporting the $\ell = \infty$ lumps

Because of translational invariance, the lumps have zero modes. If the zero mode is given an infinitesimal expectation value, the position of the lump changes infinitesimally. Such a zero mode is quite analogous to the modes associated to marginal operators in string theory. The lump zero mode alone, however, cannot implement large translations — this is the statement that large marginal deformations require giving expectation values to other fields. This was explored in some detail for string field theory in [15] and for the ϕ^3 toy model in [8].

In particular the focus in [15] and [8] was on a special phenomenon. In string field theory, the effective potential for the marginal parameter is only defined up to a critical value of this parameter [15]. An explanation for this was proposed in [8]. The idea is that as the displacement of the lump increases from zero to infinity, the marginal parameter first grows, reaches a maximum and then decreases down to zero. Higher level fields take over in displacing the lump as the displacement grows.

Evidence for this proposal was given in the ϕ^3 model, but there the role of the higher level fields was taken by the continuum of scalar fields. In the present $\ell \to \infty$ model there is no continuum, and we can see explicitly how the higher level fields, all corresponding to localized bound states do the job of moving the lump. Thus, the $\ell \to \infty$ model is expected to model more closely the behavior of marginal parameters in string field theory.

We can see this for the $\ell \to \infty$ model by decomposing the shifted lump into the fluctuation modes of the unshifted lump. The shifted lump has a profile

$$\overline{\phi}(x - x_0) = \exp\left(-\frac{(x-x_0)^2}{4}\right) = \sum_{n=0}^{\infty} A_n \psi_n(x)\,, \tag{4.1}$$

where $\psi_n(x)$ are the harmonic oscillator wavefunctions. In this notation the A_n's are the fields living on the lump. The wavefunctions, given by

$$\psi_n(x) = \frac{1}{2^{n/2}\sqrt{n!}} H_n\left(\frac{x}{\sqrt{2}}\right) e^{-x^2/4}\,, \tag{4.2}$$

are normalized to

$$\int dx \psi_n(x)\psi_m(x) = \sqrt{2\pi}\,\delta_{mn}\,. \tag{4.3}$$

Since $\psi_0(x) = \overline{\phi}(x)$, the tachyon field T representing a fluctuation of the lump is related to A_0 as $A_0 = 1 + T$. The coefficients A_n can be found in a standard quantum mechanics text [16] and are given by

$$A_n(x_0) = \frac{{x_0}^n}{2^n\sqrt{n!}} \exp\left(-\frac{{x_0}^2}{8}\right). \tag{4.4}$$

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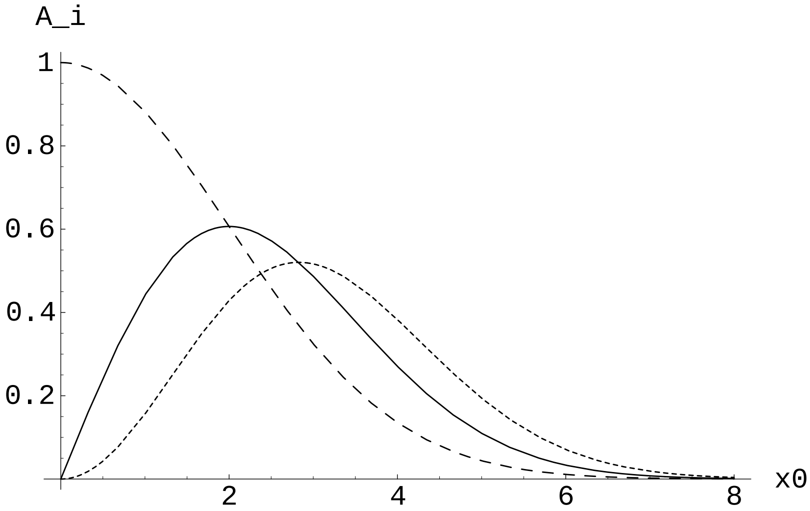

Figure 3: The dashed line is the behavior of $A_0 = 1 + T$ as a function of the lump displacement x_0. The continuous line shows the marginal mode $A_1(x_0)$, which takes a maximum value for $x_0 = 2$. The dotted line is $A_2(x_0)$, which takes a maximum value after A_1 does.

Clearly, shifting the lump turns on more than just the massless mode A_1, it turns on all fields. For small x_0, we have $A_1 \sim x_0/2$, but A_1 takes a maximum value for $x_0 = 2$:

$$A_1^{\max} = A_1(x_0 = 2) = \exp(-\frac{1}{2}) \simeq 0.6065\,. \tag{4.5}$$

$A_1(x_0)$ decreases for $x_0 > 2$ beyond. A plot showing the behavior of A_0, A_1 and A_2 as functions of x_0 is shown in figure 3. For $x_0 = 2$, the lump constructed using the four lowest modes (A_0 up to A_3) is centered at about $x_0 = 1.9$. Using only A_0 and A_1 the lump ends up centered at $x = 1$.

Examining the n-dependence of the coefficients A_n for large x_0 one can show that they become largest for $n \sim x_0^2/4$. Since the level L of a field is simply given as $L(n) = n$ (see (2.23)), we see that that in displacing the lump a large distance x_0, fields of level

$$L = \frac{x_0^2}{4}\,, \tag{4.6}$$

provide the largest contribution.

We have attempted to reproduce the above qualitative behavior by constructing the effective potential of the tachyon and marginal mode only. Let α denote the zero mode amplitude and T denote the tachyon amplitude. In other words, we set

$$\phi(x) = (1 + T + \alpha\, x) \exp\left(-\frac{x^2}{4}\right)\,. \tag{4.7}$$

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Note that $\alpha = A_1$ since $\psi_1(x) = x\exp(-x^2/4)$. The effective potential for these modes is

$$V_{\rm eff}(\alpha,T) = \int dx \bigg\{ (1+T)^2\frac{x^2}{8} + \frac{\alpha^2}{8}\left(1-\frac{x^2}{2}\right)^2 - \qquad (4.8)$$
$$- \frac{1}{4}(1+T+\alpha x)^2\left(\ln(1+T+\alpha x)^2 - \frac{1}{2}x^2\right)\bigg\}\exp\left(-\frac{x^2}{2}\right).$$

For a given α we want to find the T that minimizes $V_{\rm eff}$. In all previous models, one finds that the equation of motion of T cannot be satisfied for values of α greater than a critical value. Taking the T derivative of (4.8) gives

$$\frac{dV_{\rm eff}}{dT} = -\frac{1}{2}\int dx(1+T+\alpha x)\ln(1+T+\alpha x)^2\exp\left(-\frac{x^2}{2}\right). \qquad (4.9)$$

The expression in (4.9) is clearly zero if $T=-1$. This corresponds to the expectation value that erases the lump. There can be another zero for a value of $T \in [-1,0]$. In fact, for small α one can convince oneself that (4.9) has a zero for T slightly below zero. In general this second solution exists as long as $V_{\rm eff}''|_{T=-1} < 0$.

Hence, the critical value α is defined by the condition that $V_{\rm eff}''|_{T=-1} = 0$. Taking another derivative on (4.9), we find

$$\frac{d^2V_{\rm eff}}{dT^2}\bigg|_{T=-1} = -\int dx\left[1+\frac{1}{2}\ln(\alpha x)^2\right]\exp\left(-\frac{x^2}{2}\right)$$
$$= -\sqrt{2\pi}\left(1 - \frac{\gamma+\ln 2}{2} + \ln\alpha\right), \qquad (4.10)$$

where γ is the Euler constant. Hence, the maximum value for α is reached when

$$\alpha = \alpha_{\rm max} = \sqrt{2}\exp(-1+\gamma/2) \approx 0.6943\,, \qquad (4.11)$$

in decent agreement with (4.5). We note, however, that for $\alpha_{\rm max}$ we do not get in this model a reasonable picture of a displaced lump. Indeed, the marginal branch starts at $T=0$ for $\alpha=0$ and flows to $T=-1$ at $\alpha=\alpha_{\rm max}$. For values of $\alpha \leq 0.6$ the resulting field configuration resembles that of a displaced lump, but not at the endpoint of the branch, as can be seen in (4.7). This, we believe is an intrinsic limitation of this two-field model.

There is, in addition, the branch where $T=-1$, for any α. It corresponds to fluctuations along the zero mode of the lump around the stable vacuum. The marginal branch merges into this branch for the critical value of α.

5. Higher codimension lumps and descent relations in the $\ell = \infty$ model

The energy density of the codimension one brane defined by the lump solution (a p-brane) is given by

$$E = \int d^p\vec{y}\, dx \left[\frac{1}{2}\left(\frac{d\overline{\phi}}{dx}\right)^2 + V(\overline{\phi})\right] = (\mathrm{Vol}_y) \int dx \left(\frac{d\overline{\phi}}{dx}\right)^2 . \tag{5.1}$$

With the lump profile $\overline{\phi}(x) = \exp(-x^2/4)$ one readily finds

$$T_p = \frac{E}{(\mathrm{Vol}_y)} = \frac{1}{4}\sqrt{2\pi}\,, \tag{5.2}$$

for the tension of the p-brane. Given that the original unstable vacuum $\phi_0^2 = e^{-1}$ is supposed to represent the space-filling $(p+1)$-brane, we have $T_{p+1} = V(\phi_0) = 1/(4e)$. Therefore

$$\frac{1}{2\pi}\frac{T_p}{T_{p+1}} = \frac{e}{\sqrt{2\pi}} \simeq 1.084\,, \tag{5.3}$$

a ratio that in string theory takes the value of unity.

We now consider higher codimension lumps and show that (i) their profiles can be found exactly, and (ii) the ratio in (5.3) holds for all values of p. Let c denote the value of the codimension ($c = 1$ for the one dimensional lump $\overline{\phi}(x) = \exp(-x^2/4)$). The differential equation for the profile takes the form

$$\frac{d^2\phi}{d\rho^2} + \left(\frac{c-1}{\rho}\right)\frac{d\phi}{d\rho} - V'(\phi) = 0\,, \tag{5.4}$$

where $\rho = (x_1^2 + \cdots x_c^2)^{1/2}$ is the radial coordinate transverse to the brane. The relevant solution of this differential equation is easily found:

$$\overline{\phi}(\rho) = \exp\left(-\frac{\rho^2}{4} + \frac{1}{2}(c-1)\right). \tag{5.5}$$

Indeed the value $\overline{\phi}(0)$ at the core of the lump increases from one, the value for $c = 1$, as the codimension is increased. This is in accord with the intuition that the second term in (5.4), in the mechanical analogy of motion in a potential $-V$, represents a friction term that is overcome by letting the field at the core have a larger expectation value. Higher codimension lumps exist, despite Derrick's no-go theorem, because the potential V is not bounded below.

Having the expression for the lump we can calculate its tension. Letting E_c denote the energy of the codimension c lump, we have

$$E(c) = (\mathrm{Vol}_y) \int d^c x \left[\frac{1}{2}\left(\frac{d\overline{\phi}}{d\rho}\right)^2 + V(\overline{\phi})\right]. \tag{5.6}$$

This gives, for the tension $T(c)$ of the codimension c brane

$$T(c) \equiv \frac{E(c)}{\mathrm{Vol}_y} = \frac{1}{4}\, e^{c-1} \int d^c x[\rho^2 - (c-1)]e^{-\rho^2/2} \,. \tag{5.7}$$

Using $\int d^c x = \mathrm{Vol}(S_{c-1}) \int_0^\infty d\rho \rho^{c-1}$, with $\mathrm{Vol}(S_{c-1}) = \frac{2\pi^{c/2}}{\Gamma(c/2)}$, the integral is readily done and gives:

$$T(c) = \frac{1}{4}\, e^{c-1}(2\pi)^{c/2} \,. \tag{5.8}$$

This result, for $c = 0$ agrees with the value $T_{p+1} = 1/(4e)$, and for $c = 1$ agrees with the value of T_p in (5.2). Finally, we see that it implies that the ratio in (5.3) is the same for any pair of branes whose codimension differs by one unit. One can show that the spectrum of fields living on the codimension c lump is governed by the Schroedinger potential of the c-dimensional simple harmonic oscillator.

6. Adding gauge fields

In this section we consider a possible scenario for gauge fields in our model. For components along the brane, there seems to be a neat generalization of the gauge fixed action [17] for all values of ℓ. For components perpendicular to the brane, the generalization does not appear to be as nice, although it improves in the limit $\ell \to \infty$.

The gauge fixed string field action is consistent because there is a BRST symmetry. Our philosophy is to include terms in the action that are similar to terms found in gauge fixed string field theory and save questions about BRST invariance for future work. Another possibility would be to try to obtain a gauge invariant formulation of the models. Experience in string field theory, however, suggests that this may require the inclusion of auxiliary fields [18].

6.1 The dynamics of the gauge field on the brane

In string field theory, the gauge-fixed action has the cubic term of the form

$$\phi A_\mu A^\mu \,. \tag{6.1}$$

In level zero string field theory, ϕ is proportional to $V''(\phi) - V''(0)$, since $V(\phi)$ is cubic. Hence, a possible generalization to the mass term in (6.1) is

$$\beta\,(V''(\phi) - V''(\phi_0))A_\mu A^\mu \,, \tag{6.2}$$

where ϕ_0, representing the open string vacuum, is at the maximum of the potential and β is a constant. At the very least, the gauge fields should be massless when

$\phi = \phi_0$. For the solvable ℓ models, we choose $\beta = -\frac{1}{2}\frac{\ell}{\ell+2}$, in which case the term in (6.2) can be reexpressed as

$$\mathcal{L}_{\ell+1} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}\frac{V_{\ell+1}{}'(\phi)}{\phi} A_\mu A^\mu + \cdots\,, \tag{6.3}$$

where we have included the gauge fixed kinetic term for the vector field. Note that for the ϕ^3 (V_3) model, the coefficient β differs from the corresponding coefficient in the string field theory [17].

Now consider A_μ fluctuations, where μ is an index along the brane. To this end, we write $A_\mu = \mathcal{A}_\mu(y)\eta(x)$ where y refers to coordinates on the brane worldvolume. Given the interaction term in (6.2), the masses of the gauge fluctuations are found by solving the eigenvalue equation

$$-\frac{d^2\,\eta(x)}{dx^2} + \frac{V_{\ell+1}{}'(\overline{\phi}(x))}{\overline{\phi}(x)}\,\eta(x) = m^2\,\eta(x)\,. \tag{6.4}$$

Using the explicit solutions in (2.10) and (2.5), (6.4) becomes

$$-\frac{d^2\,\eta(x)}{dx^2} + \frac{1}{2\ell}\left[\ell^2 - \ell(\ell+1)\,\mathrm{sech}^2\left(\frac{x}{\sqrt{2\ell}}\right)\right]\,\eta(x) = m^2\,\eta(x)\,. \tag{6.5}$$

Hence, just as in the scalar field case, the fluctuations satisfy the Schroedinger equation for a reflectionless potential. However, in the gauge case, this potential is U_ℓ and not $U_{\ell+1}$. Hence, using (2.4) and rescaling, the possible discrete values for the masses are

$$\begin{aligned} m^2 &= \frac{1}{2\ell}\left[\ell^2 - (\ell-p)^2\right], \quad 0 \le p < \ell \\ &= \frac{(n-1)(2\ell-n+1)}{2\ell}, \quad 1 \le n < \ell+1\,. \end{aligned} \tag{6.6}$$

Thus, except for the absent tachyon, the gauge fields have a spectrum that is the same as the scalar fields. Without the tachyon, the lowest level fluctuation is massless.

In the $\ell \to \infty$ limit, the term in (6.3) reduces to

$$\frac{1}{4}\left(\ln\phi^2 + 1\right) A_\mu A^\mu\,. \tag{6.7}$$

The mass squared levels are equally spaced and match up with the scalar field mass levels. Again, the lowest state is massless. Clearly, when ϕ rolls to the closed string vacuum, the mass of the gauge field diverges.

6.2 Giving a mass to the gauge field transverse to the brane

One of the more mysterious questions in open string field theory is what happens to gauge field components that are polarized transverse to the brane. There cannot be zero modes associated with them, since as we have seen in the previous sections, the zero mode is accounted for by a zero mode in the scalar field.

In the gauge fixed action there are, in addition to the term discussed in the previous subsection, terms of the form [17]

$$(\partial_\mu\partial_\nu\phi)\, A_\mu A_\nu\,. \tag{6.8}$$

In our toy models, we will assume that there is a similar term, except that ϕ is replaced with $-\ln\phi$. Note for the $\ell\to\infty$ model, $-\ln\phi$ is quite natural, since ${V_\infty}'(\phi)/\phi = -\ln\phi - 1/2$. For the finite ℓ models, the $\ln\phi$ term seems more *ad hoc*. Nor can we justify this term by gauge invariance. We simply assume that this belongs to the gauge fixed action.[3] Thus, the final lagrangian quadratic in the gauge fields is

$$\mathcal{L}_{\ell+1} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}\frac{{V_{\ell+1}}'(\phi)}{\phi}A_\mu A^\mu + (\partial_\mu\partial_\nu \ln\phi)\, A^\mu A^\nu\,. \tag{6.9}$$

The term in (6.8) *does* change the mass spectrum for the modes of the transversely polarized gauge field. Using the lump solution (2.10) we find

$$\frac{d^2}{dx^2}\ln\overline{\phi}(x) = -\frac{1}{2}\,\mathrm{sech}^2\left(\frac{x}{\sqrt{2\ell}}\right). \tag{6.10}$$

Hence, on account of (6.9) we find that for the transverse components equation (6.5) is modified to

$$-\frac{d^2\eta(x)}{dx^2} + \frac{1}{2\ell}\left[\ell^2 - \ell(\ell-1)\,\mathrm{sech}^2\left(\frac{x}{\sqrt{2\ell}}\right)\right]\eta(x) = m^2\eta(x)\,. \tag{6.11}$$

So again we find the Schroedinger equation for a reflectionless potential, but now this is the $U_{\ell-1}$ potential. Hence the masses of the bound modes are

$$\begin{aligned} m^2 &= \frac{1}{2\ell}\left[\ell^2 - (\ell-1-p)^2\right], \quad 0\le p<\ell-1 \\ &= \frac{(n-1)(2\ell-n+1)}{2\ell}, \quad 2\le n<\ell+1\,. \end{aligned} \tag{6.12}$$

Hence the mass spectrum is the same as for the scalar fluctuations, except now the tachyon *and* the massless mode are missing.

In the $\ell\to\infty$ limit, after replacing $-\ln\phi - 1/2$ by $V'(\phi)/\phi$, we see that the relative coefficient between the last two terms in (6.9) is a factor of two. Examining the gauge-fixed string field action in [17], one finds the same relative coefficient between the terms.

[3] The string field action has other terms quadratic in the gauge fields, although these extra terms all have $\partial_\mu A^\mu$ factors. One can not really get rid of these terms by going to a Lorentz gauge, since in principle this action is already gauge fixed. These extra terms seem to ruin the solvability of the gauge fluctuations. Without any further justification, we do not include these types of terms.

7. Concluding remarks

In this paper we have considered toy models for open string field theory with actions

$$\mathcal{S}_{\ell+1} = \int d^D x \Big\{ -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_{\ell+1}(\phi) - \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \\ - \frac{1}{2}\frac{{V_{\ell+1}}'(\phi)}{\phi} A_\mu A^\mu + (\partial_\mu\partial_\nu \ln\phi)\, A^\mu A^\nu \Big\}, \tag{7.1}$$

where

$$V_{\ell+1}(\phi) = \frac{\ell}{4}\phi^2\left(1-\phi^{2/\ell}\right). \tag{7.2}$$

In the limit that $\ell \to \infty$, the action in (7.1) simplifies to

$$\mathcal{S}_\infty = \int d^D x \Big\{ -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}\phi^2\ln\phi^2 - \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \\ + \frac{1}{4}\left(\ln\phi^2+1\right) A_\mu A^\mu + (\partial_\mu\partial_\nu \ln\phi)\, A^\mu A^\nu \Big\}. \tag{7.3}$$

The $\ell = \infty$ model in (7.3) appears to be more stringy than the finite ℓ models or the $\ell = 3$ model studied in [8]. First, the continuum spectrum is absent. Second, upon tachyon condensation all modes disappear from the spectrum. Third, mass levels are equally spaced and the tachyon on the lump has the same mass as the tachyon in the original field theory model. Fourth, higher codimension lumps satisfy simple descent relations.

On the other hand the $\ell = \infty$ model also has some features that are less similar to those in open string field theory. In some sense $\ell = 3$ matches quite nicely with the cubic nature of open string field theory, and with the level zero tachyon action. In addition, tachyon condensation in $\ell = 3$ involves most of the spectrum just as in string field theory, while in $\ell = \infty$ only the tachyon mode condenses. Finally, being non polynomial, and having a level zero tachyon field, it is not completely obvious how to deal with the level expansion in the $\ell = \infty$ model. In this respect, the model seems quite similar to closed string field theory [19]. Learning how to do level expansion in the $\ell = \infty$ theory may eventually help to understand the still mysterious fate of the closed string tachyon.

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